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Stabilization of the non-trivial relative equilibria of a gyrostat with an elastic element in a circular orbit☆

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Abstract

The motion of a gyrostat, regarded as a rigid body, in a circular Kepler orbit in a central Newtonian force field is investigated in a limited formulation. A uniformly rotating statically and dynamically balanced flywheel is situated in the rigid body. A uniform elastic element, which, during the motion of the system, is subjected to small deformations, is rigidly connected to the rigid body-gyrostat body. The problem is discretized without truncating the corresponding infinite series, based on a modal analysis or using a certain specified system of functions, for example, of the assumed forms of the oscillations, which depend on the spatial coordinates and which satisfy appropriate boundary-value problems of the linear theory of elasticity. The elastic element is specified in more detail (a rod, plate, etc.), as well as its mass and stiffness characteristics and the form of the fastening, and the choice of the system of functions is determined. Non-trivial relative equilibria of the system (the state of rest with respect to an orbital system of coordinates when the elastic element is deformed) is sought approximately on the basis of a converging iteration method, described previously. It is shown, using Routh's theorem, that by an appropriate choice of the gyrostatic moment and when certain conditions, imposed on the system parameters are satisfied, one can stabilize these equilibria (ensure that they are stable).

It is well known that the stabilization of a spacecraft in an orbit requires the least amount of energy expenditure when it is stabilized in one of possible positions of relative equilibrium. This fact governs the importance of problems related to finding relative equilibria and investigating the conditions for them to be stable. The widely used model of an actual spacecraft is a gyrostat, which is a rigid body with statically and dynamically balanced flywheels situated in it, with different forms of elastic elements connected to it. The presence of trivial relative equilibria of the apparatus – states of rest with respect to an orbital system of coordinates for undeformed elements, is verified by substituting into the equilibrium equations zero values of the variables of the problem, which define the deformation, and is an exceptional case. Published results of research are mainly devoted to investigating the stability of a gyrostat with elastic elements in such an equilibrium (see, for example, Ref. 1).

In more general cases it is necessary to be able to find non-trivial relative equilibria, at least approximately (we are dealing here with numerical modelling), and to investigate the conditions for them to be stable, which will depend on the equilibria obtained. In such research, discretization of the problem,[2] by which we mean representation of the displacement vector of an arbitrary point of an elastic section of the system as a result of deformation in the form of an infinite series in a certain specified system of functions, which depend on the spatial coordinates with unknown

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coefficients which depend on time, becomes necessary, since it is determined by modern analytical methods of solving boundary-value problems of partial differential equations (Fourier's method, the Rayleigh-Ritz method, etc.). As a rule, after discretization an a priori reduction of the problem is carried out: in the infinite series indicated a finite number of terms is retained from the very beginning. The problems of finding non-trivial equilibria and investigating their stability has thereby been reduced to the same problems but for a mechanical system with a finite number of degrees of freedom (see, for example, Ref. 3).

Such a procedure, although it was constrained and determined by existing methods of investigating such problems, has often been criticised (Ref. 4, p. 120), since it did not give explicit criteria for choosing the final number of terms of the infinite series, retained on truncation, and if the truncation was not carried out, it did not give explicit criteria for the positive definiteness of the corresponding quadratic form of an infinite number of variables.

In this paper, using discretization of the problem and without truncation of the corresponding infinite series based on Routh's theorem,[5,6] we investigate the possibility of stabilizing non-trivial equilibria about an orbital system of coordinates of the system considered when it moves in a circular Kepler orbit in a central Newtonian field of attractive forces by choosing the gyrostatic moment and satisfying certain conditions, imposed on the other system parameters.

It should be noted that the motion in a circular Kepler orbit of a gyrostat with a uniform rectilinear elastic rod, the axis of which, in its natural state, is situated in some principal central plane of inertia of the gyrostat was considered previously in Ref.7 in the same force field as considered here and with similar assumptions in a limited formulation; in this case it was possible to obtain exact analytical expressions defining two single-parameter families of non-trivial equilibrium uniaxial orientations about the attracting centre of the gyrostat with an elastic rod and the conditions for them to be stable. In that paper, general speaking, it turned out that one could only approximately seek the equilibria of the system about an orbital system of coordinates but in turn it was possible to obtain the previously unknown exact sufficient conditions for them to be stable. In both investigations, although touching on an investigation of different steady motions of a gyrostat with different elastic elements, the discretization of the problem without a priori truncation of the corresponding infinite series and finding and investigating the stability of the steady motions of a mechanical system with a denumerable number of degrees of freedom based on Routh's theorem, which were imposed within the framework of the direct Lyapunov method, were characteristic features.

1. Formulation of the problem

1.1. Non-trivial relative equilibria

Suppose that, in the body of the gyrostat, which is modelled by a rigid body containing a uniformly rotating statically and dynamically balanced flywheel, rigidly fastened along a certain region Γ with a non-zero measure, there is a uniform elastic section of proportional shape. During the motion the elastic section experiences small deformations. The system moves in a central Newtonian attractive force field such that its instantaneous centre of mass O is uniformly displaced along a Kepler circular orbit around the attracting centre, and ω is the orbital angular velocity of the system, where $\boldsymbol{\omega}=\omega\boldsymbol{\beta}$ and $\omega\equiv|\boldsymbol{\omega}|$. The motion of the system is considered in a limited formulation.[8]

We will introduce the following right rectangular Cartesian axes of coordinates: an orbital system of coordinates Oy_k $(k=1, 2, 3)$ with a pole at the instantaneous centre of mass of the system and unit vectors of the axes $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ respectively; the unit vector $\boldsymbol{\beta}$ is directed along the normal to the orbital plane, $\boldsymbol{\gamma}$ is directed along the radius vector of the instantaneous centre of mass with respect to the attracting centre, the trihedron $O_1x_k(k=1, 2, 3)$ with unit vectors of the axes $\mathbf{i}_k$ is rigidly connected to the body of the gyrostat, O_1 is the centre of mass of the undeformed system, while the coordinate axes coincide with its principal central axes of inertia. Suppose $\boldsymbol{\Omega}$ is the angular velocity of the coupled system of coordinates O_1x_k with respect to the orbital system of coordinates, v_2 is the region occupied by points of the undeformed elastic element, v_1 is the region of points of the gyrostat, $v=v_1+v_2$; m_1 is the mass of the gyrostat, m_2 is the mass of the elastic section, ρ is the density of the masses and $m=m_1+m_2$.

The radius vector with respect to the centre of mass O of an arbitrary point of the system, defined before deformation with respect to the point O_1 by the vector $\mathbf{r}$, after deformation will be given by the expression $(\mathbf{r}+\mathbf{u}(t, \mathbf{r})-\mathbf{r}_0)$, where $\mathbf{u}(t, \mathbf{r})$ is the elastic displacement vector, $\mathbf{r}_0=m^{-1}\int_{\upsilon_2}\rho\mathbf{u}(t, \mathbf{r})d\upsilon$ is the radius vector of the instantaneous centre of mass O with respect to O_1 and $\mathbf{u}(t, \mathbf{r})=0$ when $\mathbf{r}\in\upsilon_1$. Henceforth we will neglect the quantity $\mathbf{r}_0$, i.e. the points O_1 and O coincide.

We will formulate propositions for which further discussion will be carried out. (Everywhere henceforth, unless otherwise stated, the subscripts n, m and p take values 1, 2, ...; summation over a subscript is carried out from unity to infinity, and integration is carried out over the region υ.)

1°. We will represent the elastic displacement vector as follows:

$$\mathbf{u}(t,\mathbf{r}) = \sum_n \tilde{q}_n(t)\tilde{\boldsymbol{\varphi}}_n(\mathbf{r}), \tag{1.1}$$

where the unknown quantities $\tilde{q}_n(t)$ must be regarded as Lagrangian coordinates, which define the deformation of the elastic section; the specified functions $\tilde{\boldsymbol{\varphi}}_n(\mathbf{r})$ satisfy certain homogeneous equations of the theory of elasticity, to be specific, the boundary conditions of the rigid clamping of the elastic element to the surface Γ and the absence of distributed forces and moments on its free surface. Moreover, we will assume that the functions $\tilde{\varphi}_n$ are orthonormalized:

$$(\tilde{\boldsymbol{\varphi}}_n, \tilde{\boldsymbol{\varphi}}_m) \equiv \int_{\upsilon_2} \rho\tilde{\boldsymbol{\varphi}}_n\tilde{\boldsymbol{\varphi}}_m d\upsilon = \delta_{nm}$$

and $\tilde{\boldsymbol{\varphi}}_n(\mathbf{r}) = 0$ when $\mathbf{r} \in \upsilon_1$. For example,[7] in the case when the elastic element is a rod, we can use Krylov's beam functions as $\tilde{\boldsymbol{\varphi}}_n(\mathbf{r})$.

2°. The potential energy of elastic deformations is given by the expression

$$\Pi = \frac{1}{2}\sum_{n,p}\tilde{c}_{np}\tilde{q}_n\tilde{q}_p, \quad \tilde{c}_{np} = \Lambda_n^2\delta_{np}, \quad 0 < \Lambda_1 < \Lambda_2 < \ldots, \tag{1.2}$$

where, in turn, the quantities Λ are determined using the solutions of the corresponding transcendental equation (compare with the situation considered previously in Ref. 7). For our further analysis it will be extremely fruitful to introduce the new variables $q_n(t) \equiv (\tilde{c}_{nn})^{1/2}\tilde{q}_n(t)$ and, in accordance with representation (1.1), the new functions $\boldsymbol{\varphi}_n(\mathbf{r}) \equiv (\tilde{c}_{nn})^{-1/2}\tilde{\boldsymbol{\varphi}}_n(\mathbf{r})$, $(\boldsymbol{\varphi}_n, \boldsymbol{\varphi}_m) = \Lambda_n^{-2}\delta_{nm}$. It follows from the natural, physically justified assumption that the energy of elastic deformations is limited, that $q(t) \equiv (q_1, q_2, \ldots)$ belongs to a Hilbert space l_2 of infinite sequences,[9] bounded in norm $||q|| \equiv \left(\sum_n |q_n|^2\right)^{1/2}$.

3°. The potential energy of gravitational forces (apart from a known constant) is given by the expression

$$\Pi_g = \frac{1}{2}\omega^2(3\boldsymbol{\gamma}\mathbf{J}\boldsymbol{\gamma} - \operatorname{tr}\mathbf{J}), \tag{1.3}$$

where, when expression (1.1) and the formula for q_n are taken into account, the inertia tensor of the system with respect to its centre of mass has the form

$$\mathbf{J}(q) \equiv \int \rho((\mathbf{r}+\mathbf{u})^2 E - (\mathbf{r}+\mathbf{u}) : (\mathbf{r}+\mathbf{u}))d\upsilon = \mathbf{I}_0 + \sum_n q_n\mathbf{J}_n + \sum_{n,m} q_nq_m\mathbf{J}_{nm}.$$

Here and henceforth $E(E_\infty)$ is the identity matrix of dimension 3×3 $(\infty \times \infty)$, and the colon denotes the diad product of vectors.

The inertia tensor of the undeformed system $\mathbf{I}_0$ ($I_0 = \operatorname{diag}(I_0^1, I_0^2, I_0^3)$ is its matrix of the components in the coupled system of coordinates) and the tensors $\mathbf{J}_n$ and $\mathbf{J}_{nm}$ can be written as follows:

$$\mathbf{I}_0 \equiv \int \rho(\mathbf{r}^2 E - \mathbf{r} : \mathbf{r})d\upsilon, \quad \mathbf{J}_n \equiv \int \rho(2\mathbf{r}\boldsymbol{\varphi}_n E - \mathbf{r} : \boldsymbol{\varphi}_n - \boldsymbol{\varphi}_n : \mathbf{r})d\upsilon$$

$$\mathbf{J}_{nm} \equiv \int \rho(\boldsymbol{\varphi}_n\boldsymbol{\varphi}_m E - (\boldsymbol{\varphi}_n : \boldsymbol{\varphi}_m + \boldsymbol{\varphi}_m : \boldsymbol{\varphi}_n)/2)d\upsilon$$

where $I_n = [I_n^{ij}]$, $I_{nm} = [I_{nm}^{ij}]$ are symmetrical matrices of the components of the tensors $\mathbf{J}_n$ and $\mathbf{J}_{nm}$ in the coupled system of coordinates.

4°. The central ellipsoid of inertia of the undeformed system is not a figure of revolution.

The equations of motion of the system in the case considered, which can be obtained by different methods,[2] are not used here. It is well known that, in addition to the integrals of the direction cosines $U_i (i = 1, 2, 3)$, they allow of a Jacobi type integral U. We have

$$\begin{aligned} &U_1 \equiv \boldsymbol{\gamma\gamma} - 1 = 0, \quad U_2 \equiv \boldsymbol{\beta\beta} - 1 = 0, \quad U_3 \equiv \boldsymbol{\gamma\beta} = 0 \\ &U \equiv T_r + \Pi + \Pi_g - 1/2\boldsymbol{\omega}\mathbf{J}(q)\boldsymbol{\omega} - \boldsymbol{\omega}\mathbf{k} = \text{const}, \end{aligned} \tag{1.4}$$

where T_r is the kinetic energy of relative motion of the system with a stopped flywheel[1] and $\mathbf{k}$ is the gyrostatic moment. When $\mathbf{\Omega} = 0$, $\dot{q}(t) \equiv (\dot{q}_1, \dot{q}_2 \ldots) = 0$ the quantity T_r vanishes; we have the estimate

$$\exists \varepsilon_T > 0 : T_r > \varepsilon_t\left(\mathbf{\Omega}^2 + \sum_n \dot{q}_n^2\right).$$

The dot denotes a derivative with respect to time.

To find the relative equilibria of the system we will use Routh's theorem (see also Refs. 10,11). We will introduce a functional – a combination of the changed potential energy of the system and of the integrals of the direction cosines

$$V_1(\boldsymbol{\gamma}, \boldsymbol{\beta}, q, \lambda, \sigma, \nu) \equiv \left(\Pi + \Pi_g - \frac{1}{2}\boldsymbol{\omega}\mathbf{J}(q)\boldsymbol{\omega} - \mathbf{k}\boldsymbol{\omega}\right) + 3\omega^2\lambda U_3 + \omega^2\nu U_3 - \frac{3}{2}\omega^2\sigma U_1,$$

where λ, σ, ν are undetermined Lagrange multipliers. Equating the first variation of the functional V_1 to zero, we obtain the following system of equations for determining the relative equilibria of the system (here and henceforth $\hat{\mathbf{\Omega}} = 0, \hat{q} = 0$):

$$(\mathbf{J}(\hat{q}) - \sigma E)\hat{\boldsymbol{\gamma}} + \lambda\hat{\boldsymbol{\beta}} = 0 \Leftrightarrow \sigma = \hat{\boldsymbol{\gamma}}\mathbf{J}(\hat{q})\hat{\boldsymbol{\gamma}}, \quad \lambda = -\hat{\boldsymbol{\beta}}\mathbf{J}(\hat{q})\hat{\boldsymbol{\gamma}}, \quad \hat{\boldsymbol{\alpha}}\mathbf{J}(\hat{q})\hat{\boldsymbol{\gamma}} = 0 \tag{1.5}$$

$$\begin{aligned} &(\nu E - \mathbf{J}(\hat{q}))\hat{\boldsymbol{\beta}} + 3\lambda\hat{\boldsymbol{\gamma}} - \boldsymbol{\eta} = 0 \Leftrightarrow \nu = \hat{\boldsymbol{\beta}}\mathbf{J}(\hat{q})\hat{\boldsymbol{\beta}} + \hat{\nu} \\ &\hat{\boldsymbol{\alpha}}\mathbf{J}(\hat{q})\hat{\boldsymbol{\beta}} = -\hat{\boldsymbol{\alpha}}\boldsymbol{\eta}, \quad \hat{\boldsymbol{\gamma}}\mathbf{J}(\hat{q})\hat{\boldsymbol{\beta}} = -\hat{\boldsymbol{\gamma}}\boldsymbol{\eta}/4 \end{aligned} \tag{1.6}$$

$$\hat{q}_n + \omega^2(3\hat{\boldsymbol{\gamma}}\mathbf{J}_n'(\hat{q})\hat{\boldsymbol{\gamma}} - \operatorname{tr}\mathbf{J}_n'(\hat{q}) - \hat{\boldsymbol{\beta}}\mathbf{J}_n'(\hat{q})\hat{\boldsymbol{\beta}})/2 = 0. \tag{1.7}$$

We have introduced the following notation

$$\hat{\nu} \equiv \hat{\boldsymbol{\beta}}\boldsymbol{\eta}, \quad \boldsymbol{\eta} \equiv \mathbf{k}/\omega, \quad \mathbf{J}_n' \equiv \mathbf{J}_n + 2\sum_p q_p\mathbf{J}_{np}.$$

The variables with a hat define the perturbed motion – the relative equilibrium of the system; small perturbations of the corresponding variables are denoted by $\delta\mathbf{\Omega}$, $\delta\dot{q} \equiv (\delta\dot{q}_1, \delta\dot{q}_2, \ldots)$ etc.

To obtain the non-trivial positions of relative equilibrium of the system, which will be determined by the existence of non-zero solutions of the denumerable system of equations (1.7),[12] the gyrostatic moment $\mathbf{k}$ must be chosen so that its projections onto the $\boldsymbol{\alpha}$, $\boldsymbol{\gamma}$ axes of the orbital system of coordinates is equal to zero, i.e. in Eqs. (1.5) and (1.6) $\hat{\boldsymbol{\gamma}}\boldsymbol{\eta} = \hat{\boldsymbol{\alpha}}\boldsymbol{\eta} = 0$. The projection of the gyrostatic moment onto the normal to the orbital plane, defined with respect to the coupled system of coordinates by the direction cosines of the unit vector $\hat{\boldsymbol{\beta}}$, remains so far undetermined – the quantity $\hat{\nu} \equiv \hat{\boldsymbol{\beta}}\boldsymbol{\eta}$ is not determined. This can be used to ensure stability of the relative equilibria. A similar situation is also typical in problems of the stability of the relative equilibria of a gyrostat without an elastic element.

If we assume that non-trivial relative equilibria of the system exist, then, in the projections onto the principal central axes of inertia of the system ($\mathbf{e}_j(\hat{q})$ are the unit vectors of the corresponding axes, $j = 1, 2, 3$, while the matrix of the components of the tensor $\mathbf{J}(\hat{q})$ in these axes is diagonal: $J = \operatorname{diag}(J_1^0, J_2^0, J_3^0)$), constructed for this equilibrium, we will have[12]

$$\begin{aligned} &\hat{\gamma} \equiv (\hat{\gamma}_1, \hat{\gamma}_2, \hat{\gamma}_3), \quad \hat{\gamma}_i = \pm\delta_{im}; \quad \sigma \equiv \hat{\gamma}J\hat{\gamma}^T = J_0^m \\ &\hat{\beta} \equiv (\hat{\beta}_1, \hat{\beta}_2, \hat{\beta}_3), \quad \hat{\beta}_i = \pm\delta_{ik}; \quad \nu \equiv J_0^k + \hat{\beta}_k|\boldsymbol{\eta}|, \quad \lambda = 0, \quad \boldsymbol{\eta}\hat{\boldsymbol{\alpha}} = \boldsymbol{\eta}\hat{\boldsymbol{\gamma}} = 0, \\ &\forall k, m \in \{1, 2, 3\}, \quad k \neq m, \end{aligned} \tag{1.8}$$

i.e. in the relative equilibrium the unit vectors $\hat{\boldsymbol{\beta}}$ and $\hat{\boldsymbol{\gamma}}$ are collinear with the unit vectors $\mathbf{e}_j(\hat{q})$, while the quantities $\hat{q}_n$ will be found from the equations

$$\hat{q}_n + \omega^2\left[3\left(J_n^{mm} + 2\sum_p \hat{q}_p J_{np}^{mm}\right) - \operatorname{tr} J'_n - J_n^{kk} - 2\sum_p \hat{q}_p J_{np}^{kk}\right]/2 = 0. \tag{1.9}$$

Here $J_n = [j_n^{ij}]$, $J_{np} = [J_{np}^{ij}]$, etc. are symmetrical matrices of the components of the corresponding tensors in the $\{\mathbf{e}_j\}$ axes ($i, j = 1, 2, 3$), i.e. the matrices of the components of the tensors are denoted by the same letters but without being distinguished by heavy type.

It is extremely difficult to obtain an exact solution of Eqs. (1.5)–(1.7). A method for approximately finding non-trivial relative equilibria for a system consisting of a rigid body (a gyrostat with a stopped flywheel) and an arbitrary uniform elastic section was proposed in Ref. 12. It is completely applicable in the case being considered: approximately, in the form of a power series $\hat{q}_n$, one finds the eigenvalues $J_0^j(\hat{q})$ and the eigen unit vectors $\mathbf{e}_i(\hat{q})$ of the tensor $\mathbf{J}(\hat{q})$

$$J_0^j(\hat{q}) = I_0^j + \sum_n \hat{q}_n I_n^{jj} + \dots, \quad \mathbf{e}_j(\hat{q}) = \mathbf{i}_j + \sum_n \hat{q}_n\left(\mathbf{i}_m \frac{I_n^{jm}}{I_0^j - I_0^m} + \mathbf{i}_k \frac{I_n^{jk}}{I_0^j - I_0^k}\right) + \dots \tag{1.10}$$

$$\forall j \in \{1, 2, 3\}, \quad \forall m, k \in \{1, 2, 3\}, \quad m \neq k \neq j.$$

Knowing the expressions for $\mathbf{e}_j$, the matrix of the transition of $P(\hat{q})$ from the coupled system of coordinates with unit vectors $\{\mathbf{i}_j\}$ to a system of coordinates with unit vectors $\{\mathbf{e}_j(\hat{q})\}$ can be written as follows:

$$P(\hat{q}) \equiv (e_1(\hat{q}), e_2(\hat{q}), e_3(\hat{q})) = E + \sum_n \hat{q}_n P_n + \dots,$$

where the skew-symmetric matrix $P_n = [(\delta_{ij} - 1)I_n^{ij}/(I_0^i - I_0^j)]$ and $\mathbf{e}_j(\hat{q})$ is the column of components of $\mathbf{e}_j$ in the coupled system of coordinates ($i, j = 1, 2, 3$).

On the basis of the theorem on the conversion of matrices of the components of tensors on changing to another system of coordinates, we can write $J_n = P(\hat{q}) I_n P^T(\hat{q})$, $J_{np} = P I_{np} P^T$, etc. Neglecting terms that are non-linear in $\hat{q}_n$ in Eq. (1.9), we obtain the following linearized system for finding $\hat{q}_n$

$$\hat{q}_n + \omega^2 \sum_p \left(2I_{np}^{mm} - 2I_{np}^{kk} - I_{np}^{ll} + 4\frac{I_n^{km} I_p^{km}}{I_0^k - I_0^m} + 3\frac{I_n^{lm} I_p^{lm}}{I_0^l - I_0^m} + \frac{I_n^{kl} I_p^{kl}}{I_0^k - I_0^l}\right)\hat{q}_p = I_n^{kk} - I_n^{mm} + \frac{I_n^{ll}}{2}. \tag{1.11}$$

Here and henceforth k and m are already fixed in according with the choice of $l \in \{1, 2, 3\}$, $l \neq k \neq m$ in relations (1.8).

We propose to solve the infinite system of Eq. (1.11) by the reduction method,[9] as regards the convergence of which the corresponding assertion is proved in Ref. 12; in this case, if a normalization is first carried out, the solution of system (1.11) will belong to the unit sphere in l_2, i.e. $||\hat{q}|| \leq 1$.

We will now introduce the infinite matrix

$$C = [\delta_{np}/\omega^2 + 2I_{np}^{mm} - 2I_{np}^{kk} - I_{np}^{ll}] \tag{1.12}$$

and we will require it to be positive definite

$$\exists \varepsilon_C > 0 : \forall q \in l_2 \quad qCq^T \geq \varepsilon_C \|q\|^2 \tag{1.13}$$

(below, in Section 2, we will present the conditions which ensure that this requirement is satisfied).

Now the assertion that the reduction method converges, which reduces to the successive solution of "truncated" equations (1.11) (i.e. when, for fixed N ($n = 1, 2, 3, \ldots N$) summation over p in relations (1.11) is carried out from 1 to N and $N \to \infty$) can be formulated as follows.[12]

Assertion 1 *(.).* The reduction method for solving the infinite system of equations (1.11) will be convergent if the infinite sequence $\{\Lambda_n^{-1}\} \in l_2$, condition (1.13) is satisfied and

$$\min_{i,\, j \in \{1, 2, 3\},\, i \neq j} \left|I_0^i - I_0^j\right| \geq (\varepsilon_C \pi)^{-1} \left(\sum_{n,\, p} \left(4\left|I_n^{km} I_p^{km}\right| + 3\left|I_n^{lm} I_p^{lm}\right| + \left|I_n^{kl} I_p^{kl}\right|\right)^2 \right)^{1/2}, \tag{1.14}$$

where the arbitrary quantity $\pi \in (0, 1)$.

Note that the series on the right-hand side of inequality (1.14) converges, for example, when $\{\Lambda_n^{-1}\} \in l_2$ and this condition can be satisfied by an appropriate choice of the moments of inertia of the gyrostat itself.

2. The conditions for stabilizing non-trivial relative equilibria of the system

By Routh's theorem, the requirements imposed on the parameters of the problem and which ensure, when there are linear constraints which restrict the perturbations, that the second variation $\delta^2 V_1(0)$ of the functional V_1, calculated for the unperturbed motion will be positive definite on the equilibrium (1.8) and (1.9), will guarantee its stability in Lyapunov's sense (taking into account the fact that T_r is positive definite) with respect to the norm for the perturbations $(\delta\boldsymbol{\Omega}\delta\boldsymbol{\Omega} + ||\delta\dot{q}||^2 + \delta\boldsymbol{\gamma}\delta\boldsymbol{\gamma} + \delta\boldsymbol{\beta}\delta\boldsymbol{\beta} + ||q||^2)^{1/2}$.

Suppose, for convenience, that

$$w_1 \equiv \delta\gamma_1, \quad w_2 \equiv \delta\beta_1, \quad w_3 \equiv \delta\gamma_2, \quad w_4 \equiv \delta\beta_2, \quad w_5 \equiv \delta\gamma_3, \quad w_6 \equiv \delta\beta_3; \quad w \equiv (w_1, \ldots, w_6).$$

We will assume that $(w, \delta q) \in l_2$. The linear constraints, imposed on the perturbations, are obtained from the conditions

$$\begin{aligned} \delta U_1(0) &\equiv \hat{\gamma}_1 w_1 + \hat{\gamma}_2 w_3 + \hat{\gamma}_3 w_5 = 0 \\ \delta U_2(0) &\equiv \hat{\beta}_1 w_2 + \hat{\beta}_2 w_4 + \hat{\beta}_3 w_6 = 0 \\ \delta U_3(0) &\equiv \hat{\beta}_1 w_1 + \hat{\gamma}_1 w_2 + \hat{\beta}_2 w_3 + \hat{\gamma}_2 w_4 + \hat{\beta}_3 w_5 + \hat{\gamma}_3 w_6 = 0. \end{aligned} \tag{2.1}$$

The second variation of the functional can be written as follows (see Ref. 13):

$$\delta^2 V_1(0) = \omega^2 (w, \delta q) \begin{Vmatrix} A & B \\ B^T & C \end{Vmatrix} (w, \delta q)^T, \tag{2.2}$$

where, provided that the tensors and vectors are specified by their own components in the $\{\mathbf{e}_i(\hat{q})\}$ axes,

$$A \equiv \mathrm{diag}(3(J_0^1 - \sigma), (\nu - J_0^1), 3(J_0^2 - \sigma), (\nu - J_0^2), 3(J_0^3 - \sigma), (\nu - J_0^3)), \tag{2.3}$$

and the square matrix B consists of six rows $b_i \equiv b_{i1}, b_{i2}, \ldots (i = 1, \ldots 6$ of infinite length, and their components are

$$\begin{aligned} b_{2j-1,\, n} &= 3\left(J_n^{jm} + 2\sum_p \hat{q}_p J_{np}^{jm}\right)\hat{\gamma}_m \\ b_{2j,\, n} &= -\left(J_n^{jk} + 2\sum_p \hat{q}_p J_{np}^{jk}\right)\hat{\beta}_k; \quad j = 1, 2, 3. \end{aligned} \tag{2.4}$$

The matrix C is identical with that introduced above (formula (1.12)).

Using the Cauchy inequality, the expressions for the matrices J_n and J_{np} of the components of the tensors $\mathbf{J}_n$ and $\mathbf{J}_{np}$ and the conditions for normalizing the functions $\boldsymbol{\varphi}_n(\mathbf{r})$, it can be shown that $b_i \in l_2 (i = 1, \ldots, 6)$ when $\{\Lambda_n^{-1}\} \in l_2$, $\hat{q} \in l_2$.

For the matrix C, defined by formula (1.12), the following assertion has been proved in Ref. 13.

Assertion 2. The bounded quadratic form of the variables from l_2 with matrix C will be positive definite, i.e. condition (1.13) will be satisfied if the infinite sequence $\{\Lambda_n^{-1}\} \in l_1$ and the following inequality is satisfied

$$\omega^{-2} > 3\Lambda_1^{-2} + 8\Lambda_1^{-1} \sum_{p=2}^{\infty} \Lambda_p^{-1}. \tag{2.5}$$

Note that we can take as ε_C from condition (1.13) any quantity which satisfies the inequalities

$$0 < \varepsilon_C < \omega^{-2} - \left(3\Lambda_1^{-2} + 8\Lambda_1^{-1} \sum_{p=2}^{\infty} \Lambda_p^{-1}\right).$$

Suppose the arbitrary quantity $\varepsilon \in (0, \varepsilon_C)$. Expression (2.2) can be represented as follows:

$$\omega^{-2}\delta^2 V_1(0) = w(A - \varepsilon BB^T)w^T + \delta q(C - \varepsilon^2 E_\infty)\delta q^T + (\varepsilon^{-1} wB + \varepsilon\delta q)^2.$$

It can be seen that, for $\delta^2 V_1(0)$ to be positive definite it is sufficient to satisfy the requirements of Assertion 2, which was a positive definite quadratic form of the variables w with matrix $(A - \varepsilon^{-2}BB^T)$ when the linear constraints (2.1), imposed on w (Ref. 14) are satisfied.

We will introduce the following quantities

$$\Delta \equiv 3(J_0^l - J_0^m) - d_{11} > 0$$

$$a \equiv 4(J_0^k - J_0^m) - d_{22} - d_{12}^2/\Delta, \quad b \equiv J_0^k - J_0^l - d_{33} - d_{13}^2/\Delta, \quad c \equiv d_{23} + d_{12}d_{13}/\Delta,$$

where

$$d_{ij} \equiv \varepsilon^{-2} d_i d_j^T, \quad d_i \equiv (d_{i1}, d_{i2}, \ldots), \quad i, j = 1, 2, 3.$$

Here the components of the infinite rows

$$d_{1n} \equiv 3\left(J_n^{lm} + 2\sum_p \hat{q}_p J_{np}^{lm}\right), \quad d_{2n} \equiv 4\left(J_n^{km} + 2\sum_p \hat{q}_p J_{np}^{km}\right), \quad d_{3n} \equiv J_n^{kl} + 2\sum_p \hat{q}_p J_{np}^{kl}.$$

It is obvious that $d_i \in l_2 (i = 1, 2, 3)$, as also the rows $b_1, \ldots, b_6$.

We can now state an assertion on the stabilization of non-trivial relative equilibria of the system as the conditions which ensure their stability. Omitting the intermediate, quite clear by lengthy calculations we obtain the following assertion.

Assertion 3. In order for the non-trivial relative equilibrium of system (1.8), (1.9) to be stable, it is sufficient to satisfy the following conditions

$$\{\Lambda_n^{-1}\} \in l_1, \quad \omega^{-2} > 3\Lambda_1^{-2} + 8\Lambda_1^{-1} \sum_{p=2}^{\infty} \Lambda_p^{-1} \tag{2.6}$$

$$\Delta \equiv 3(J_0^l - J_0^m) - d_{11} > 0 \tag{2.7}$$

$$|\boldsymbol{\eta}|\hat{\beta}_k \equiv \hat{\nu} > -(4(J_0^k - J_0^m) - d_{22}) + d_{12}^2/\Delta \tag{2.8}$$

and simultaneously with the condition (2.8)

$$2\hat{\nu} > -(a+b) + ((b-a)^2 + 4c^2)^{1/2} \text{ или } 2\hat{\nu} < -(a+b) - ((b-a)^2 + 4c^2)^{1/2}. \tag{2.9}$$

Note that all the quantities occurring in expressions (2.6)–(2.9), are finite, while $J_0^i (i = 1, 2, 3)$ are the moments of inertia about the $\{\mathbf{e}_i(\hat{q})\}$ axes “frozen” in the non-trivial equilibrium of the system. Obviously, the moments of inertia of the gyrostat occur in these quantities, by an appropriate choice of which we can ensure that condition (2.7) is satisfied.

This can easily be proved if we use as $J_0^i (i = 1, 2, 3)$, for example, representation (1.10) and we make estimates similar to those made previously in Ref. 7.

Conditions (2.8) and (2.9) can be satisfied by an appropriate choice of the gyrostatic moment, the components of which in the coupled system of coordinates are obtained using the matrix $P(\hat{q})$.

Condition (2.6) impose constraints on the stiffness of the elastic element and the properties of the function $\{\boldsymbol{\varphi}_n(\mathbf{r})\}$. By increasing the stiffness of the elastic element, the inequality in (2.6) can also be satisfied. For example,[7] if the elastic element of the system is a rod and its flexural deformations correspond to Kirchhoff's hypotheses, then $\Lambda_n^2 = EI\beta_n^4/\rho$, where β_n are the corresponding roots of the equation

$$\cos\beta\,\mathrm{ch}\,\beta + 1 = 0$$

(it can be shown that $\{\Lambda_n^{-1}\} \in l_1$)[12] and EI is the flexural stiffness of the rod. As mentioned earlier the Krylov beam functions are used as $\boldsymbol{\varphi}_n(\mathbf{r})$.

As $\Lambda_1 \to \infty$ (then $d_{ij} \to 0$ $(i, j = 1, 2, 3)$) relations (2.6)–(2.9) become the well-known conditions for the stability of relative equilibria of the gyrostat, when the gyrostatic moment is situated on one of the principal central axes of its ellipsoid of inertia and is directed along the normal to the orbital plane.[14]

It should be noted that conditions (2.6)–(2.9) are more rigorous than the conditions for stability of the system, which are "hardened" in a position of relative equilibrium, when we must formally put $d_{ij} = 0$ in them and completely reject conditions (2.6). The destabilizing effect of elastic elements in problems of the stability of relative equilibria of complex mechanical systems has been discussed earlier in Refs. 1,2,4,15.

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